Exercise 28

Solve the differential equation using the method of variation of parameters.

$$y'' + 4y' + 4y = \frac{e^{-2x}}{x^3}$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 4y_c' + 4y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} + 4(re^{rx}) + 4(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 4r + 4 = 0$$

Solve for r.

$$(r+2)^2 = 0$$

$$r = \{-2\}$$

Two solutions to the ODE are e^{-2x} and xe^{-2x}. By the principle of superposition, then,

$$y_c(x) = C_1 e^{-2x} + C_2 x e^{-2x}.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' + 4y_p' + 4y_p = \frac{e^{-2x}}{x^3} \tag{2}$$

In order to obtain a particular solution, use the method of variation of parameters: Allow the parameters in the complementary solution to vary.

$$y_p = C_1(x)e^{-2x} + C_2(x)xe^{-2x}$$

Differentiate it with respect to x.

$$y_p' = C_1'(x)e^{-2x} + C_2'(x)xe^{-2x} - 2C_1(x)e^{-2x} - C_2(x)(2x-1)e^{-2x}$$

If we set

$$C_1'(x)e^{-2x} + C_2'(x)xe^{-2x} = 0, \tag{3}$$

then

$$y_p' = -2C_1(x)e^{-2x} - C_2(x)(2x-1)e^{-2x}.$$

Differentiate it with respect to x once more.

$$y_p'' = -2C_1'(x)e^{-2x} - C_2'(x)(2x-1)e^{-2x} + 4C_1(x)e^{-2x} + 4C_2(x)(x-1)e^{-2x}$$

Substitute these formulas into equation (2).

$$\left[-2C_1'(x)e^{-2x} - C_2'(x)(2x-1)e^{-2x} + \cancel{4C_1(x)e^{-2x}} + \cancel{4C_2(x)(x-1)e^{-2x}}\right] + 4\left[\cancel{-2C_1(x)e^{-2x}} - \cancel{C_2(x)(2x-1)e^{-2x}}\right] + 4\left[\cancel{C_1(x)e^{-2x}} + \cancel{C_2(x)xe^{-2x}}\right] = \frac{e^{-2x}}{x^3}$$

Simplify the result.

$$-2C_1'(x)e^{-2x} - C_2'(x)(2x-1)e^{-2x} = \frac{e^{-2x}}{x^3} \tag{4}$$

Multiply both sides of equation (3) by 2, and multiply both sides of equation (4) by 1.

$$2C_1'(x)e^{-2x} + 2C_2'(x)xe^{-2x} = 0$$

$$-2C_1'(x)e^{-2x} - C_2'(x)(2x-1)e^{-2x} = \frac{e^{-2x}}{x^3}$$

Add the respective sides of these equations to eliminate $C_1'(x)$.

$$C_2'(x)e^{-2x} = \frac{e^{-2x}}{x^3}$$

Solve for $C_2'(x)$.

$$C_2'(x) = \frac{1}{x^3}$$

Integrate this result to get $C_2(x)$, setting the integration constant to zero.

$$C_2(x) = -\frac{1}{2x^2}$$

Solve equation (3) for $C_1'(x)$.

$$\begin{aligned} C_1'(x) &= -C_2'(x)x \\ &= -\left(\frac{1}{x^3}\right)x \\ &= -\frac{1}{x^2} \end{aligned}$$

Integrate this result to get $C_1(x)$, setting the integration constant to zero.

$$C_1(x) = \frac{1}{x}$$

Therefore,

$$\begin{aligned} y_p &= C_1(x)e^{-2x} + C_2(x)xe^{-2x} \\ &= \left(\frac{1}{x}\right)e^{-2x} + \left(-\frac{1}{2x^2}\right)xe^{-2x} \\ &= \frac{e^{-2x}}{2x}, \end{aligned}$$

and the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1e^{-2x} + C_2xe^{-2x} + \frac{e^{-2x}}{2x}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.